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INTERPOLATION OF GIBBS MEASURES WITH WHITE NOISE FOR HAMILTONIAN PDE

TADAHIRO OH, JEREMY QUASTEL, AND BENEDEK VALKÓ

ABSTRACT. We consider the family of interpolation measures of Gibbs measures and white noise given by

$$dQ_{0,\beta}^{(p)} = Z_\beta^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-1/2}\}} e^{-\int_{\mathbb{T}} u^2 + \beta \int u^p} dP_{0,\beta}$$

where $P_{0,\beta}$ is the Wiener measure on the circle, with variance β^{-1} , conditioned to have mean zero. It is shown that as $\beta \rightarrow 0$, Q_0^β converges weakly to mean zero Gaussian white noise Q_0 . As an application, we present a straightforward proof that Q_0 is invariant for the Kortweg-de Vries equation (KdV). This weak convergence also shows that the white noise is a weak limit of invariant measures for the modified KdV and the cubic nonlinear Schrödinger equations.

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1. INTRODUCTION

1.1. An interpolation of measures. Let Q_0 denote the mean zero Gaussian white noise on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. i.e. Q_0 is the probability measure on real-valued distributions u with $\int_{\mathbb{T}} u = 0$ satisfying

$$\int e^{i\langle f, u \rangle} dQ_0(u) = e^{-\frac{1}{2}\|f\|_{L^2}^2} \quad (1.1)$$

for any mean zero smooth real-valued function f on \mathbb{T} , where $\langle \cdot, \cdot \rangle$ denotes the pairing between the Schwartz space $\mathcal{S}(\mathbb{T})$ and its dual $\mathcal{S}'(\mathbb{T})$. It is known that Q_0 is supported

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on the Sobolev space $H_0^s(\mathbb{T})$ for $s < -\frac{1}{2}$, where $H_0^s(\mathbb{T})$ consists of real-valued distributions $u = \sum_{n \neq 0} \hat{u}_n e^{2\pi i n x} \in \mathcal{S}'$ with $\hat{u}_{-n} = \overline{\hat{u}_n}$ such that $\|u\|_{H_0^s}^2 = \sum_{n \neq 0} |n|^{2s} |\hat{u}_n|^2 < \infty$.

Let P_0 denote the Wiener measure on $u \in C(\mathbb{T})$ conditioned to have $\int_{\mathbb{T}} u = 0$. It can be derived from the Brownian Bridge P as follows: For a given $x \in \mathbb{R}$, condition a standard Brownian motion $u(t)$, $t \in [0, 1]$, starting at $u(0) = x$ to have $u(1) = x$ and $\int_{\mathbb{T}} u = 0$. Then distribute $u(0)$ according to a real Gaussian with mean zero and variance $\pi^2/3$. The easiest way to check that this produces the appropriate measure is by the Fourier representation of u : Let $\{g_n\}_{n \geq 1}$ be a family of independent standard complex-valued Gaussian random variables, i.e. its real and imaginary parts are independent Gaussians with mean zero and variance $1/2$. Also, for $n \geq 1$, let $g_{-n} = \overline{g_n}$. Then

$$u(x) = \sum_{n \neq 0} \frac{g_n}{n} e^{2\pi i n x}. \quad (1.2)$$

Similarly, let $P_{0,\beta}$ be the Wiener measure with variance β^{-1} conditioned to have $\int_{\mathbb{T}} u = 0$. Formally, we can write $P_{0,\beta}$ as

$$dP_{0,\beta} = Z_{0,\beta}^{-1} \exp\left(-\frac{\beta}{2} \int_{\mathbb{T}} u_x^2\right) \prod_{x \in \mathbb{T}} du(x). \quad (1.3)$$

and under $P_{0,\beta}$,

$$u(x) = \tilde{\beta}^{-1/2} \sum_{n \neq 0} \frac{g_n}{n} e^{2\pi i n x}, \quad \tilde{\beta} = 4\pi^2 \beta. \quad (1.4)$$

For fixed $K > 0$ and $p \in \mathbb{N}$, let $P_0^{\varphi_1^p}$ denote the probability measure on $u \in C(\mathbb{T})$ with $\int_{\mathbb{T}} u = 0$ given by

$$dP_0^{\varphi_1^p} = Z_{p,K}^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\}} e^{\int u^p} dP_0. \quad (1.5)$$

The L^2 -cutoff is necessary to make the normalization $Z_{p,K}$ well-defined and finite (for $p \leq 6$ [LRS, B2].) The notation φ_1^p is borrowed from quantum field theory; the superscript p denotes the order of the nonlinearity and the subscript the dimension. The measure $P_0^{\varphi_1^p}$ corresponds to the Gibbs measure for certain Hamiltonian PDEs. We will discuss this aspect in the next subsection.

We can also define a family of probability measures depending on $\beta > 0$,

$$dP_{0,\beta}^{\varphi_1^p} = \hat{Z}_{\beta}^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-1/2}\}} e^{\beta \int u^p} dP_{0,\beta}, \quad (1.6)$$

where $\hat{Z}_{\beta} = \hat{Z}(\beta, p, K)$. Finally, let $Q_{0,\beta}^p$, $\beta > 0$, be the following family of probability measures on $u \in C(\mathbb{T})$ with $\int_{\mathbb{T}} u = 0$, interpolating between $P_{0,\beta}^{\varphi_1^p}$ and Q_0 ;

$$dQ_{0,\beta}^{(p)} = Z_{\beta}^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-1/2}\}} e^{-\frac{1}{2} \int_{\mathbb{T}} u^2 + \beta \int u^p} dP_{0,\beta}. \quad (1.7)$$

In the following, we assume $p = 3$ or 4 . It follows from [LRS, B2] that for each fixed $\beta > 0$, $Q_{0,\beta}^{(p)}$ is a well-defined probability measure on $H^s(\mathbb{T})$, $s < \frac{1}{2}$, the regularity being inherited from Brownian motion on \mathbb{T} .

The main result of this article is

Theorem 1.1. *Let $p = 3$ or 4 and $K > \frac{1}{2}$. Then, as $\beta \rightarrow 0$, $Q_{0,\beta}^{(p)}$ converges weakly to Q_0 as probability measures on $H_0^s(\mathbb{T})$, $s < -\frac{1}{2}$.*

Remark 1.2. When $p = 4$, the analogue to Theorem 1.1 holds for the measures on complex-valued distributions u (without the mean zero assumption),

$$d\mathbf{Q}_\beta^{(4)} = \tilde{Z}_\beta^{-1} \mathbf{1}_{\{\int |u|^2 \leq K\beta^{-1/2}\}} e^{-\frac{1}{2} \int |u|^2 + \beta \int |u|^4} d\mathbf{P}_\beta, \quad (1.8)$$

where \mathbf{P}_β is the complex Wiener measure with variance β^{-1} . We present the proof of Theorem 1.1 in details for the real-valued case and indicate the modification for the complex-valued case.

Formally, the theorem follows from the observation that

$$\begin{aligned} dQ_{0,\beta}^{(p)} &= \bar{Z}_\beta^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-1/2}\}} e^{-\frac{1}{2} \int_{\mathbb{T}} u^2 + \beta \int u^p - \frac{\beta}{2} \int u_x^2} \prod_{x \in \mathbb{T}} du(x) \\ &\xrightarrow{\beta \rightarrow 0} \bar{Z}_0^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} u^2} \prod_{x \in \mathbb{T}} du(x) = dQ_0. \end{aligned} \quad (1.9)$$

So the result is intuitively clear. Unfortunately, neither the normalizations \bar{Z}_β nor the “flat measure” $\prod_{x \in \mathbb{T}} du(x)$ make sense, so a proof is required. It turns out to be a little tricky and it involves a careful analysis of random Fourier series.

Consider the Gaussian measure μ_β given by

$$d\mu_\beta = Z_\beta^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} u^2} dP_{0,\beta} = \hat{Z}_\beta^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} u^2 - \frac{\beta}{2} \int u_x^2} \prod_{x \in \mathbb{T}} du(x), \quad (1.10)$$

where u is real-valued with $\int_{\mathbb{T}} u = 0$.¹ This is an interpolation of the Wiener measure $P_{0,\beta}$ and the white noise Q_0 on \mathbb{T} . If u is distributed according to μ_β , then it can also be represented as

$$u(x) = \sum_{n \neq 0} \frac{g_n}{\sqrt{1 + \tilde{\beta} n^2}} e^{2\pi i n x}. \quad (1.11)$$

The main difficulty of the proof of Theorem 1.1 lies in establishing the exponential expectation estimate:

$$\mathbb{E}_{\mu_\beta} [\mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int u^p}] = \int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int u^p} d\mu_\beta \leq C(r) < \infty \quad (1.12)$$

uniformly for small $\beta > 0$, where \mathbb{E}_{μ_β} denotes an expectation with respect to μ_β . Recall that for each $\beta > 0$, u is almost surely in $H^s \setminus H^{\frac{1}{2}}$, $s < \frac{1}{2}$. However, when $\beta = 0$, (1.10) reduces to the white noise Q_0 supported on $H^s \setminus H^{-\frac{1}{2}}$, $s < -\frac{1}{2}$. Hence, $\int u^p$, $p = 3, 4$, diverges as $\beta \rightarrow 0$, and thus we need to carefully analyze $\beta \int u^p$ as $\beta \rightarrow 0$. It turns out that the decay of β and the growth of $\int u^p$ is in perfect balance when $p = 4$, (see Remark 2.2) and the proof (1.12) is much more delicate when $p = 4$. We need some probabilistic tools such as the hypercontractivity of the Ornstein-Uhlenbeck semigroup. We present the proof in the remaining sections of the article.

When $p = 4$, one can also consider the convergence of $\tilde{Q}_{0,\beta}^{(4)}$ whose density is given by

$$d\tilde{Q}_{0,\beta}^{(4)} = Z_\beta^{-1} e^{-\beta \int_{\mathbb{T}} u^4} d\mu_\beta.$$

In this case, thanks to the negative sign in front of $\beta \int_{\mathbb{T}} u^4$, we have the exponential expectation estimate (1.12) for free.

¹In the following, we use Z_β to denote various normalization constants.

Theorem 1.3. *As $\beta \rightarrow 0$, $\tilde{Q}_{0,\beta}^{(4)}$ converges weakly to Q_0 as probability measures on $H_0^s(\mathbb{T})$, $s < -\frac{1}{2}$.*

In proving Theorem 1.3, we follow the basic argument for Theorem 1.1. However, since there is no need for an L^2 -cutoff, a slight care is required. When $p = 3$, we still need an L^2 -cutoff in view of transformation $u \rightarrow -u$.

Before we proceed to the proof of Theorem 1.1, let us discuss the motivation for studying this problem and present an application to some Hamiltonian PDEs in the remaining part of this section.

1.2. Hamiltonian dynamics and Gibbs measures. Given a Hamiltonian flow on \mathbb{R}^{2n} :

$$\begin{cases} \dot{p}_i = \frac{\partial H}{\partial q_j} \\ \dot{q}_i = -\frac{\partial H}{\partial p_j} \end{cases} \quad (1.13)$$

with Hamiltonian $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$, Liouville's theorem states that the Lebesgue measure on \mathbb{R}^{2n} is invariant under the flow. Then, it follows from the conservation of the Hamiltonian H that the Gibbs measures $e^{-H(p,q)} \prod_{j=1}^n dp_j dq_j$ are invariant under the flow of (1.13).

In the context of the nonlinear Schrödinger equations (NLS) on \mathbb{T} :

$$iu_t - u_{xx} \pm |u|^{p-2}u = 0, \quad u(0) = 0, \quad (1.14)$$

Lebowitz-Rose-Speer [LRS] considered the Gibbs measure of the form

$$d\mu = \exp(-H(u)) \prod_{x \in \mathbb{T}} du(x), \quad (1.15)$$

where $H(u)$ is the Hamiltonian given by $H(u) = \frac{1}{2} \int |u_x|^2 \pm \frac{1}{p} \int |u|^p dx$. It was shown that such Gibbs measure μ is a well-defined probability measure on $H^s \setminus H^{\frac{1}{2}}$, $s < \frac{1}{2}$. (In the focusing case (with $-$), the result only holds for $p < 6$ with the L^2 -cutoff $\mathbf{1}_{\{\int |u|^2 \leq K\}}$ for any $K > 0$, and for $p = 6$ with sufficiently small K .) Using the Fourier analytic approach, Bourgain [B2] continued the study and proved the invariance of the Gibbs measure μ under the flow of NLS. In the same paper, he also established the invariance of the Gibbs measures for the Korteweg-de Vries equation (KdV) on \mathbb{T} :

$$u_t + u_{xxx} - 6uu_x = 0, \quad u(0) = u_0, \quad (1.16)$$

and the modified KdV equation (mKdV) on \mathbb{T} :

$$u_t + u_{xxx} \mp u^2 u_x = 0, \quad u(0) = u_0. \quad (1.17)$$

Invariant Gibbs measures μ for Hamiltonian PDEs can be regarded as stationary measures for infinite dimensional dynamical systems, and it follows from Poincaré recurrence theorem that almost all the points of the phase space are stable according to Poisson, i.e. if \mathcal{S}_t denotes a flow map: $u_0 \mapsto u(t) = \mathcal{S}_t u_0$, then for almost all u_0 , there exists a sequence $\{t_n\}$ tending to ∞ such that $\mathcal{S}_{t_n} u_0 \rightarrow u_0$. We also know such dynamics is also multiply recurrent in view of Furstenberg [F]: let A be any measureable set with $\mu(A) > 0$. Then, for any integer $k > 1$, there exists $n \neq 0$ such that $\mu(A \cap \mathcal{S}_n A \cap \mathcal{S}_{2n} A \cap \dots \cap \mathcal{S}_{(k-1)n} A) > 0$. Note that this recurrence property holds only in the support of the Gibbs measure, i.e. not for smooth functions.

Now note that if $F(p, q)$ is any function that is conserved under the flow of (1.13), then the measure $d\mu_F = e^{-F(p,q)} \prod_{j=1}^n dp_j dq_j$ is invariant. Recall that NLS, KdV, and mKdV

are all Hamiltonian partial differential equations preserving the L^2 -norm (see also [DLT] for another intriguing connection.) Hence, it is natural, at least at a heuristic level, to expect the invariance of the white noise for these equations. The difficulty here is the low regularity of the phase space.

1.3. Invariance of white noise for KdV on \mathbb{T} . As an application of Theorem 1.1, we present a straightforward proof of the fact that Q_0 is an invariant measure for KdV on \mathbb{T} . Given a smooth initial condition $u_0 : \mathbb{T} \rightarrow \mathbb{R}$, we have a solution $\mathcal{S}_t u_0 = u(t)$ for $-\infty < t < \infty$. In fact, KdV is well-posed for much rougher initial data; the nonlinear solution map \mathcal{S}_t extends to a continuous group of nonlinear evolution operators

$$\bar{\mathcal{S}}_t : H_0^s(\mathbb{T}) \rightarrow H_0^s(\mathbb{T}), \quad -\infty < t < \infty, \quad s \geq -1. \quad (1.18)$$

By the Fourier restriction method, Bourgain [B1] proved $s \geq 0$, and Kenig-Ponce-Vega [KPV] and Colliander et al. [CKSTT] pushed it down to $s \geq -\frac{1}{2}$. Finally, Kappeler and Topalov [KT] proved $s \geq -1$ via the inverse spectral method. Since the white noise Q_0 is supported on $H_0^s(\mathbb{T})$ for $s < -\frac{1}{2}$, this means that it makes sense to start KdV on the circle with white noise as initial data, for almost every realization.

In [QV] and [O1, O2], we proved the following result:

Theorem 1.4. *White noise Q_0 is invariant under KdV. i.e. for any $t \in \mathbb{R}$, $\bar{\mathcal{S}}_t^* Q_0 = Q_0$.*

Here, $\bar{\mathcal{S}}_t^* Q_0$ denotes the pushforward of the measure Q_0 by the map $\bar{\mathcal{S}}_t$. The proof in [QV] is indirect: We show that Q_0 is the image under the Miura transform of the Gibbs measure for the defocusing mKdV (with the $-$ sign in (1.17)), which was proven to be invariant by Bourgain [B2]. While the proof in [O1, O2] is more direct, it relies on heavy Fourier analysis. Since the result is so simple to state, it is reasonable to ask for a straightforward proof (and such a proof has been requested of the authors.)

In the following, we give a more straightforward proof of Theorem 1.4, using Theorem 1.1, (1.18), and the following.

Proposition 1.5 (Bourgain, [B2]). *$P_{0,\beta}^{\varphi_1^3}$ defined in (1.5), $\beta > 0$, are invariant for KdV.*

Note that in [B2] this is only explicitly proven for $\beta = 1$. But the same proof works for all $\beta > 0$. If μ is an invariant measure of a Markov process $u(t)$ and F is a conserved quantity; $F(u(t)) = F(u(0))$, then, as long as it makes sense, $d\nu = Fd\mu$ is an invariant measure as well. The quantity $F(u) = \int_{\mathbb{T}} u^2$ is a conserved quantity for KdV and $\exp(-\frac{1}{2} \int_{\mathbb{T}} u^2) \in L^1(P_{0,\beta}^{\varphi_1^3})$. Hence it follows from Proposition 1.5 that

Corollary 1.6. *$Q_{0,\beta}^{(3)}$ defined in (1.7), $\beta > 0$, are invariant for KdV.*

To complete the proof of Theorem 1.4, we need to verify that Q_0 , the limit of invariant measures by Theorem 1.1 and Corollary 1.6, is itself invariant.

Let ϕ be any bounded continuous function on $H_0^{-1}(\mathbb{T})$. By invariance of $Q_{0,\beta}^{(3)}$ under $\bar{\mathcal{S}}_t$, we have

$$\int \phi dQ_{0,\beta}^{(3)} = \int \phi \circ \bar{\mathcal{S}}_t dQ_{0,\beta}^{(3)}.$$

Since $\bar{\mathcal{S}}_t$ is continuous on $H_0^{-1}(\mathbb{T})$, we can take $\beta \rightarrow 0$ to obtain

$$\int \phi dQ_0 = \int \phi \circ \bar{\mathcal{S}}_t dQ_0 = \int \phi d\bar{\mathcal{S}}_t^* Q_0.$$

Taking $\phi(u) = \exp(i\langle f, u \rangle)$ for smooth mean zero functions f on \mathbb{T} , we get

$$\int e^{i\langle f, u \rangle} d\bar{\mathcal{S}}_t^* Q_0 = e^{-\frac{1}{2}\|f\|_{L^2}^2}, \quad (1.19)$$

which identifies $\bar{\mathcal{S}}_t^* Q_0$ as mean zero white noise. This completes the straightforward proof of Theorem 1.4.

The reason for calling the proof straightforward is that it is a fairly direct consequence of the intuitively obvious fact (1.9). It also has the advantage, partially exploited in the next subsection, that it does not appear to rely on special properties of KdV.

Remark 1.7. The same proof shows the invariance by KdV of mean zero white noise Q_{0,σ^2} with variance σ^2 , defined by

$$\int e^{i\langle f, u \rangle} dQ_{0,\sigma^2}(u) = e^{-\frac{\sigma^2}{2}\|f\|_{L^2}^2}.$$

1.4. Formal invariance of white noise for mKdV and cubic NLS on \mathbb{T} .

The advantage of the straightforward proof of the invariance of white noise under the KdV flow presented in the previous subsection is that it does not rely on special properties of KdV. Hence, in principle, it provides a route towards invariance of white noise for related equations.

Unfortunately, Theorem 1.1 is not enough to conclude the invariance of the white noise for mKdV or cubic NLS ((1.14) with $p = 4$), since their flows are not expected to be well-defined below $H^{-\frac{1}{2}}$. Recall that mKdV and cubic NLS are scaling-critical in H^s with $s = -\frac{1}{2}$. This means that the scaling invariance (on \mathbb{R}) $u(t, x) \mapsto \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x)$ preserves the homogeneous $H^{-\frac{1}{2}}$ -norm. It is usually expected that a nonlinear PDE is not well-posed below scaling-critical regularity, and the support of the white noise is below $H^{-\frac{1}{2}}$. Nevertheless, if we lower our standards, we are able to say something. Let us define a measure μ to be *formally invariant* for a flow \mathcal{S}_t if there exist invariant measures μ_n for \mathcal{S}_t , converging weakly to μ .

Corollary 1.8. *Mean zero white noise Q_0 is formally invariant for mKdV (1.17).*

Corollary 1.9. *Complex white noise \mathbf{Q} is formally invariant for cubic NLS ((1.14) with $p = 4$, either focussing or defocussing).*

Remark 1.10. Note that it is *not* necessarily impossible to define the flows on the support of the white noise. Indeed, one may be able to define the flow of mKdV or cubic NLS just on the support of the white noise. See Bourgain [B3] for the case of the L^2 -critical defocusing cubic NLS on \mathbb{T}^2 . The Gibbs measure on \mathbb{T}^2 is supported below $L^2(\mathbb{T}^2)$. Nonetheless, Bourgain constructed a well-defined flow on its support and established the invariance of the Gibbs measure. Also, given the formal invariance, it is very natural to expect that in these models, at least \mathcal{S}_t^* has an extension to a class of measures including white noise.

Remark 1.11. The measures $Q_{0,\beta}^{(p)}$ are well defined for $2 < p < 6$, and all $\beta > 0$. Theorem 1.1 extends readily to $2 < p \leq 4$. $p = 4$ is critical, in the sense that $\beta \int_{\mathbb{T}} u^4 = \mathcal{O}(1)$ under $Q_{0,\beta}^{(4)}$ as $\beta \rightarrow 0$, while for $2 < p < 4$, $\beta \int_{\mathbb{T}} u^p = o(1)$ under $Q_{0,\beta}^{(p)}$. For $p > 4$, $\beta \int_{\mathbb{T}} u^p$ blows up. Note that one should not conclude from this that Theorem 1.1 cannot hold for $p > 4$. Indeed, it is quite plausible that it does. However, the method of proof used here does not extend beyond $p = 4$.

We conclude with some remarks on the concrete meaning of invariance vs formal invariance. Suppose that we want to start our dynamics, either KdV, mKdV, or cubic NLS, with u_0 , distributed according to white noise. One way to proceed is to consider some regularization u_0^β , $\beta > 0$, of the initial data u_0 , and solve the equation in a more classical sense, to obtain smooth solutions $u^\beta(t) = \mathcal{S}_t u_0^\beta$ at a later time. Then, we ask if for small $\beta > 0$, $u^\beta(t)$ is again approximately distributed according to white noise. Invariance of white noise means that this procedure is true regardless of the type of regularization one uses. Formal invariance means that there is at least one type of regularization which works: In our case, the regularized u_0^β is distributed according to $Q_{0,\beta}^{(4)}$.

This paper is organized as follows. In Section 2, we introduce the Wick-ordered monomials and prove a preliminary lemma. In Section 3, we present the proof of Theorem 1.1 for $p = 4$, assuming the exponential expectation estimate (1.12), which we prove in Sections 4 and 5. In Section 6, we briefly discuss the argument for the complex-valued case, the defocusing case (Theorem 1.3), and the $p = 3$ case.

2. WICK ORDERING

In this section, we perform a preliminary computation for the proof of Theorem 1.1 for $p = 4$. Recall that

$$dQ_{0,\beta}^{(4)} = Z_\beta^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{\beta \int_{\mathbb{T}} u^4} d\mu_\beta,$$

where μ_β is as in (1.10). Under μ_β , u is represented as a Fourier series (1.11), where g_n are independent standard complex Gaussians for $n > 0$ and $g_{-n} = \overline{g_n}$. We will need various moments of g_n , the following identity can be proved e.g. using the moment generating function of the complex Gaussian:

$$\mathbb{E} \left[g_n^k \overline{g_n}^\ell \right] = \delta_{k\ell} k!, \quad k, \ell \in \mathbb{Z}_+, \quad (2.1)$$

where $\delta_{k\ell} = 1$ if $k = \ell$ and $= 0$ otherwise. In particular, $\mathbb{E} [g_{i_1} g_{i_2} \dots g_{i_d}] = 0$ unless we can pair the indices i_1, \dots, i_d in a way that the sum of the two indices is zero in each pair.

In order to study the behavior of $Q_{0,\beta}^{(4)}$ as $\beta \rightarrow 0$, we divide the space into several regions. For this purpose, we introduce the Wick-ordered monomials $:u^2:_\beta$ and $:u^4:_\beta$ with parameter β :

$$:u^2:_\beta := u^2 - a_\beta, \quad (2.2)$$

$$:u^4:_\beta := u^4 - 6a_\beta u^2 + 3a_\beta^2, \quad (2.3)$$

where

$$a_\beta = \mathbb{E}_{\mu_\beta} \left[\int_{\mathbb{T}} u^2 \right] = \sum_{n \neq 0} \frac{1}{1 + \tilde{\beta} n^2}.$$

For basics on Wick products and Gaussian Hilbert spaces, see e.g. [J]. Note that $:u^k:_\beta = H_k(u; a_\beta)$, where $H(x, \sigma^2)$ is the Hermite polynomial in x of degree k with parameter σ^2 . We have

$$\beta^{\frac{1}{2}} a_\beta \rightarrow \frac{1}{2} \quad \text{as } \beta \rightarrow 0, \quad (2.4)$$

since $\beta^{\frac{1}{2}} \sum_{n \neq 0} \frac{1}{1 + \tilde{\beta} n^2} \rightarrow 2 \int_0^\infty \frac{1}{1 + 4\pi^2 x^2} dx = \frac{1}{2}$ by Riemann sum approximation. Also, by letting

$$b_\beta = \sum_{n \neq 0} \frac{1}{(1 + \tilde{\beta} n^2)^2} \quad \text{and} \quad c_\beta = \sum_{n \neq 0} \frac{1}{(1 + \tilde{\beta} n^2)^4},$$

we have $\beta^{\frac{1}{2}}b_\beta \rightarrow b_0$ and $\beta^{\frac{1}{2}}c_\beta \rightarrow c_0$ for some explicit constants $b_0, c_0 > 0$.

Lemma 2.1. *We have*

$$\mathbb{E}_{\mu_\beta} \left[\int_{\mathbb{T}} : u^2 :_\beta \right] = 0, \quad \mathbb{E}_{\mu_\beta} \left[\left(\int_{\mathbb{T}} : u^2 :_\beta \right)^2 \right] = 2b_\beta, \quad (2.5)$$

$$\mathbb{E}_{\mu_\beta} \left[\int_{\mathbb{T}} : u^4 :_\beta \right] = 0. \quad (2.6)$$

Moreover, for sufficiently small $\beta > 0$, we have²

$$\mathbb{E}_{\mu_\beta} \left[\left(\int_{\mathbb{T}} : u^4 :_\beta \right)^2 \right] \lesssim \beta^{-\frac{3}{2}}. \quad (2.7)$$

Proof. For simplicity, we use \mathbb{E} for \mathbb{E}_{μ_β} . By definition, we have $\mathbb{E}[\int_{\mathbb{T}} u^2] = a_\beta$. Also, we have

$$\mathbb{E} \left[\left(\int_{\mathbb{T}} : u^2 :_\beta \right)^2 \right] = 4\mathbb{E} \left[\left(\sum_{n \geq 1} \frac{|g_n|^2 - 1}{1 + \tilde{\beta}n^2} \right)^2 \right] = 4 \sum_{n \geq 1} \frac{\mathbb{E}[(|g_n|^2 - 1)^2]}{(1 + \tilde{\beta}n^2)^2} = 2b_\beta.$$

Using the representation of u under μ_β , we have

$$\int_{\mathbb{T}} u^4 = \sum_{\substack{n_{1234}=0 \\ n_j \neq 0}} \prod_{j=1}^4 \frac{g_{n_j}}{\sqrt{1 + \tilde{\beta}n_j^2}}, \quad (2.8)$$

where $n_{1234} := n_1 + n_2 + n_3 + n_4$. We say that we have a *pair* if we have $n_j = -n_k$, $j \neq k$ in the summation in (2.8). Under the condition $n_{1234} = 0$, we have either two pairs or no pair. Now, let $A_j = \{n_1 = -n_j\}$, $j = 2, 3, 4$. Then, by symmetry, we can express the sum in (2.8) as

$$\begin{aligned} \sum_{\substack{n_{1234}=0 \\ n_j \neq 0}} &= \sum_{\text{pair}} + \sum_{\text{no pair}} = \sum_{j=2}^4 \sum_{A_j} - \sum_{j < k} \sum_{A_j \cap A_k} + \sum_{\text{no pair}} \\ &= 3 \sum_{\substack{n_1 = -n_2, n_3 = -n_4 \\ n_j \neq 0}} - 3 \sum_{\substack{n_1 = n_3 = -n_2 = -n_4 \\ n_j \neq 0}} + \sum_{\text{no pair}}. \end{aligned} \quad (2.9)$$

(Note that $A_2 \cap A_3 \cap A_4$ is empty.) From (2.3), we have

$$\begin{aligned} \int_{\mathbb{T}} : u^4 :_\beta &= 3 \left\{ \sum_{n_1, n_3 \neq 0} \frac{|g_{n_1}|^2 |g_{n_3}|^2}{(1 + \tilde{\beta}n_1^2)(1 + \tilde{\beta}n_3^2)} - 2a_\beta \int_{\mathbb{T}} u^2 + a_\beta^2 \right\} \\ &\quad - 3 \sum_{n \neq 0} \frac{|g_n|^4}{(1 + \tilde{\beta}n^2)^2} + \sum_{\text{no pair}} \prod_{j=1}^4 \frac{g_{n_j}}{\sqrt{1 + \tilde{\beta}n_j^2}} \\ &= 12 \left(\sum_{n_1 \geq 1} \frac{|g_{n_1}|^2 - 1}{1 + \tilde{\beta}n_1^2} \right) \left(\sum_{n_3 \geq 1} \frac{|g_{n_3}|^2 - 1}{1 + \tilde{\beta}n_3^2} \right) \\ &\quad - 6 \sum_{n \geq 1} \frac{|g_n|^4}{(1 + \tilde{\beta}n^2)^2} + \sum_{\text{no pair}} \prod_{j=1}^4 \frac{g_{n_j}}{\sqrt{1 + \tilde{\beta}n_j^2}} =: 12\text{I}_1 - 6\text{I}_2 + \text{II}. \end{aligned} \quad (2.10)$$

²We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some $C > 0$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$.

Then, (2.6) follows from $\mathbb{E}[|g_n|^2 - 1]^2 = 1$ and $\mathbb{E}[|g_n|^4] = 2$. Using $\mathbb{E}[|g_n|^2 - 1]^4 = 9$, we have

$$\begin{aligned} \mathbb{E}[I_1^2] &= \sum_{\substack{n_1, n_3 \geq 1 \\ n_1 \neq n_3}} \frac{\mathbb{E}[|g_{n_1}|^2 - 1]^2}{(1 + \tilde{\beta} n_1^2)^2} \frac{\mathbb{E}[|g_{n_3}|^2 - 1]^2}{(1 + \tilde{\beta} n_3^2)^2} + \sum_{n \geq 1} \frac{\mathbb{E}[|g_n|^2 - 1]^4}{(1 + \tilde{\beta} n^2)^4} \\ &\leq \frac{b_\beta^2}{4} + \frac{9c_\beta}{2} \lesssim \beta^{-1} \end{aligned}$$

for sufficiently small $\beta > 0$. Similarly, we have $\mathbb{E}[I_2^2] \lesssim b_\beta^2 + c_\beta \lesssim \beta^{-1}$. Moreover, we have

$$\mathbb{E}[I_1 \cdot \Pi] = \mathbb{E}[I_2 \cdot \Pi] = 0. \quad (2.11)$$

by the comment after (2.1). Finally, we consider

$$\mathbb{E}[\Pi^2] = \mathbb{E} \left[\left(\sum_{\substack{n_{1234}=0 \\ n_j \neq 0 \\ \text{no pair}}} \prod_{j=1}^4 \frac{g_{n_j}}{\sqrt{1 + \tilde{\beta} n_j^2}} \right) \left(\sum_{\substack{k_{1234}=0 \\ k_j \neq 0 \\ \text{no pair}}} \prod_{j=1}^4 \frac{g_{k_j}}{\sqrt{1 + \beta k_j^2}} \right) \right].$$

Since the summation indices $\{n_j\}$ and $\{k_j\}$ contain no pair, we see that the only nonzero contribution comes from $\{n_1, n_2, n_3, n_4\} = -\{k_1, k_2, k_3, k_4\}$. Thus, we have

$$\mathbb{E}[\Pi^2] = 24 \mathbb{E} \left[\left(\sum_{*} \prod_{j=1}^4 \frac{|g_{n_j}|^2}{1 + \tilde{\beta} n_j^2} \right) \right]$$

where $*$ = $\{n_{1234} = 0, n_j \neq 0, \text{ and no pair}\}$. By separating the summation into (a) n_j all distinct, (b) $n_1 = n_2 \neq n_3, n_4$ and $n_3 \neq n_4$, and (c) $n_1 = n_2 = n_3 \neq n_4$ (up to permutations of the indices), we have

$$\mathbb{E}[\Pi^2] = 24 \left\{ \sum_{n_j, \text{ all distinct}}^{*} + 6 \cdot 2 \sum_{\substack{n_1=n_2 \neq n_3, n_4 \\ n_3 \neq n_4}}^{*} + 4 \cdot 6 \sum_{n_1=n_2=n_3 \neq n_4}^{*} \prod_{j=1}^4 \frac{1}{1 + \tilde{\beta} n_j^2} \right\}$$

since $\mathbb{E}[|g_n|^4] = 2$ and $\mathbb{E}[|g_n|^6] = 6$. From the positivity of the summands and by Riemann sum approximation, we have

$$\begin{aligned} \mathbb{E}[\Pi^2] &\lesssim \sum_{n_1, n_2, n_3 \neq 0} \prod_{j=1}^3 \frac{1}{1 + \tilde{\beta} n_j^2} \frac{1}{1 + \tilde{\beta} (n_1 + n_2 + n_3)^2} \\ &\quad + \sum_{n_1, n_3 \neq 0} \frac{1}{(1 + \tilde{\beta} n_1^2)^2} \frac{1}{1 + \tilde{\beta} n_3^2} \frac{1}{1 + \tilde{\beta} (2n_1 + n_3)^2} + \sum_{n_1, n_3 \neq 0} \frac{1}{(1 + \tilde{\beta} n_1^2)^3} \frac{1}{1 + \tilde{\beta} (3n_1)^2} \\ &\sim \tilde{\beta}^{-\frac{3}{2}} \int_{\mathbb{R}^3} \prod_{j=1}^3 \frac{1}{1 + x_j^2} \frac{1}{1 + (x_1 + x_2 + x_3)^2} dx_1 dx_2 dx_3 \\ &\quad + \tilde{\beta}^{-1} \int_{\mathbb{R}^2} \frac{1}{(1 + x_1^2)^2} \frac{1}{1 + x_3^2} \frac{1}{1 + (2x_1 + x_3)^2} dx_1 dx_3 \\ &\quad + \tilde{\beta}^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{1}{(1 + x_1^2)^3} \frac{1}{1 + (3x_1)^2} dx_1 \lesssim \beta^{-\frac{3}{2}} \end{aligned}$$

for sufficiently small $\beta > 0$. Hence, we obtain (2.7). \square

Remark 2.2. The moral is that the main contribution of $\int_{\mathbb{T}} : u^4 :_{\beta}$ comes from the “no pair, all distinct” part. From (2.6) and (2.4), we see that $\mathbb{E}[\beta \int_{\mathbb{T}} u^4] = 3\beta a_{\beta}^2 = O(1)$. This shows that the decay of β and the growth of $\int_{\mathbb{T}} u^4$ is in perfect balance.

3. PROOF OF THEOREM 1.1: $p = 4$

In order to prove Theorem 1.1, it suffices to show that, for any smooth mean 0 function f on \mathbb{T} ,

$$C_{\beta} \int e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} d\mu_{\beta} \rightarrow e^{-\frac{1}{2}\|f\|_{L^2}^2}, \quad \text{as } \beta \rightarrow 0 \quad (3.1)$$

for some $C_{\beta} > 0$. Indeed (3.1) implies

$$\int e^{i \int_{\mathbb{T}} f u} dQ_{0,\beta}^{(4)} = \frac{C_{\beta} \int e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} d\mu_{\beta}}{C_{\beta} \int e^{\beta \int_{\mathbb{T}} u^4} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} d\mu_{\beta}} \rightarrow \frac{e^{-\frac{1}{2}\|f\|_{L^2}^2}}{e^{-\frac{1}{2}\|0\|_{L^2}^2}} = e^{-\frac{1}{2}\|f\|_{L^2}^2}. \quad (3.2)$$

This means that the joint distribution of the Fourier coefficients of u under $Q_{0,\beta}^{(4)}$ converges weakly to the joint distribution of the coefficients from the white noise Q_0 . The weak convergence of $Q_{0,\beta}^{(4)}$ to Q_0 in $H_0^s(\mathbb{T})$, $s < -\frac{1}{2}$, now follows from the following lemma, whose proof is presented at the end of this section.

Lemma 3.1. *The sequence of measures $Q_{0,\beta}^{(4)}$ is tight in $H_0^s(\mathbb{T})$, $s < -\frac{1}{2}$, as $\beta \rightarrow 0$.*

It follows from Lemma 3.1 and Prohorov’s theorem that for any sequence $\{\beta_j\}$ of positive numbers tending to 0, the sequence $\{Q_{0,\beta_j}^{(4)}\}$ is sequentially compact. Moreover, by the comment after (3.2), it converges weakly to Q_0 . The same comment guarantees the uniqueness of the limit point of $\{Q_{0,\beta}^{(4)}\}$ for $\beta \rightarrow 0$. Hence, Theorem 1.1 follows.

In view of Lemma 2.1, define $\mathcal{A}_{\beta,N}$ and $\mathcal{B}_{\beta,N}$ by

$$\mathcal{A}_{\beta,N} = \left\{ \left| \int_{\mathbb{T}} : u^4 :_{\beta} \right| \leq N\beta^{-\frac{3}{4}} \right\}, \text{ and } \mathcal{B}_{\beta,N} = \left\{ \left| \int_{\mathbb{T}} : u^2 :_{\beta} \right| \leq N\beta^{-\frac{1}{4}} \right\} \quad (3.3)$$

for large N and small $\beta > 0$, and we consider separately the contributions from

$$(i) \mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}, \quad (ii) \mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}^c, \quad \text{and} \quad (iii) \mathcal{A}_{\beta,N}^c.$$

First, note that by Chebyshev’s inequality with Lemma 2.1 and (3.3), we have an easy preliminary estimate

$$\mu_{\beta}(\mathcal{A}_{\beta,N}^c \cup \mathcal{B}_{\beta,N}^c) \lesssim N^{-2}. \quad (3.4)$$

Our goal is to show that the main contribution for the weak convergence (3.1) indeed comes from (i), and that the contributions from (ii) and (iii) are small.

• (i) On $\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}$: Since $\int_{\mathbb{T}} u^4 = \int_{\mathbb{T}} : u^4 :_{\beta} + 6a_{\beta} \int_{\mathbb{T}} u^2 - 3a_{\beta}^2$ and $\int_{\mathbb{T}} : u^4 :_{\beta}$ is “small” on $\mathcal{A}_{\beta,N}$, it is natural to introduce the Gaussian probability measure

$$d\tilde{\mu}_{\beta} = Z_{\beta}^{-1} \exp\left(6\beta a_{\beta} \int_{\mathbb{T}} u^2\right) d\mu_{\beta} \quad (3.5)$$

for sufficiently small $\beta > 0$. First, we show that the normalization Z_{β} is indeed finite for (small) $\beta > 0$.

Lemma 3.2. *The normalization constant Z_β in (3.5) is bounded uniformly as $\beta \searrow 0$. Moreover,*

$$\lim_{\beta \rightarrow 0} \int e^{6\beta a_\beta \int_{\mathbb{T}} u^2} d\mu_\beta = e^{3/2}$$

Proof. From (1.11), we have, for small $\beta > 0$,

$$\begin{aligned} \int e^{6\beta a_\beta \int_{\mathbb{T}} u^2} d\mu_\beta &= \prod_{n \geq 1} \mathbb{E} \left[\exp \left(\frac{12\beta a_\beta}{1 + \tilde{\beta} n^2} |g_n|^2 \right) \right] = \prod_{n \geq 1} \frac{1}{1 - \frac{12\beta a_\beta}{1 + \tilde{\beta} n^2}} \\ &= \prod_{n \geq 1} \frac{1 + \frac{1}{\tilde{\beta} n^2}}{1 + \frac{1 - 12\beta a_\beta}{\tilde{\beta} n^2}} = \frac{\sinh(\pi \tilde{\beta}^{-\frac{1}{2}})}{\pi \tilde{\beta}^{-\frac{1}{2}}} \frac{\pi \sqrt{1 - 12\beta a_\beta} \tilde{\beta}^{-\frac{1}{2}}}{\sinh(\pi \sqrt{1 - 12\beta a_\beta} \tilde{\beta}^{-\frac{1}{2}})}. \end{aligned}$$

Here, we used $\mathbb{E}[e^{aX^2}] = (1 - 2a)^{-\frac{1}{2}}$, $a < \frac{1}{2}$, for a *real*-valued standard Gaussian random variable X , and the infinite product formula for $\sinh z$. By (2.4), we have

$$\lim_{\beta \rightarrow 0} \int e^{6\beta a_\beta \int_{\mathbb{T}} u^2} d\mu_\beta = \lim_{\beta \rightarrow 0} \exp \left(\pi(\tilde{\beta}^{-\frac{1}{2}} - \sqrt{1 - 12\beta a_\beta} \tilde{\beta}^{-\frac{1}{2}}) \right) = e^{3/2}.$$

□

Under $\tilde{\mu}_\beta$, we have

$$u(x) = \sum_{n \neq 0} \frac{g_n}{\sqrt{1 - 12\beta a_\beta + \tilde{\beta} n^2}} e^{2\pi i n x}. \quad (3.6)$$

From (2.4), we have $12\beta a_\beta \sim \beta^{\frac{1}{2}} \rightarrow 0$ as $\beta \rightarrow 0$, so this is well defined if β is small enough. The following lemma, combined with the argument following (3.1), shows that the Fourier coefficients under $\tilde{\mu}_\beta$ converge in distribution to those of the white noise.

Lemma 3.3. *There exists $C_\beta, \tilde{C}_\beta > 0$ such that*

$$\lim_{\beta \rightarrow 0} C_\beta \int e^{i \int_{\mathbb{T}} f u + 6\beta a_\beta \int_{\mathbb{T}} u^2 - 3\beta a_\beta^2} d\mu_\beta = \lim_{\beta \rightarrow 0} \tilde{C}_\beta \int e^{i \int_{\mathbb{T}} f u - 3\beta a_\beta^2} d\tilde{\mu}_\beta = e^{-\frac{1}{2} \|f\|_{L^2}^2}, \quad (3.7)$$

for any smooth mean 0 function f on \mathbb{T} ,

Proof. By a direct computation, we have

$$\begin{aligned} \int e^{i \int_{\mathbb{T}} f u} d\tilde{\mu}_\beta &= \exp \left\{ i \sum_{n \neq 0} \frac{\hat{f}_n g_n}{\sqrt{1 - 12\beta a_\beta + \tilde{\beta} n^2}} \right\} \\ &= \exp \left\{ -\frac{1}{2} \sum_{n \neq 0} \frac{|\hat{f}_n|^2}{1 - 12\beta a_\beta + \tilde{\beta} n^2} \right\} \rightarrow e^{-\frac{1}{2} \|f\|_{L^2}^2}. \end{aligned}$$

Then, (3.7) follows from $e^{-3\beta a_\beta^2} \rightarrow e^{-3/4}$ as $\beta \rightarrow 0$. □

Next, we show that $\beta \int_{\mathbb{T}} u^4$ is very close to $a_\beta \int_{\mathbb{T}} u^2$ in this case and that it does not affect the weak convergence in Lemma 3.3. For conciseness of the presentation, let us define, for a function F on $C(\mathbb{T})$,

$$I_f(F) = \int F(u) e^{i \int_{\mathbb{T}} f u + 6\beta a_\beta \int_{\mathbb{T}} u^2 - 3\beta a_\beta^2} d\mu_\beta.$$

Lemma 3.4. *Let $K > \frac{1}{2}$. Then, for $N > 0$, we have*

$$\limsup_{\beta \rightarrow 0} \left| \int_{\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} d\mu_{\beta} - I_f(1) \right| \lesssim N^{-1}. \quad (3.8)$$

Proof. On $\mathcal{A}_{\beta,N}$, we have $|e^{\beta \int_{\mathbb{T}} u^4 : \beta} - 1| \lesssim \beta^{\frac{1}{4}} N$ for $\beta \leq N^{-4}$. Hence, we have

$$\begin{aligned} & \left| \int_{\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} d\mu_{\beta} - I_f(\mathbf{1}_{\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}}) \right| \\ & \leq e^{6\beta^{\frac{1}{2}} a_{\beta} K - 3\beta a_{\beta}^2} \int |e^{\beta \int_{\mathbb{T}} u^4 : \beta} - 1| d\mu_{\beta} \lesssim \beta^{\frac{1}{4}} N. \end{aligned}$$

since $6\beta^{\frac{1}{2}} a_{\beta} K - 3\beta a_{\beta}^2 = O(1)$. Moreover, on $\mathcal{B}_{\beta,N}$, given $\varepsilon > 0$, there exists $\beta_0 > 0$ such that

$$\int_{\mathbb{T}} u^2 = \int_{\mathbb{T}} : u^2 :_{\beta} + a_{\beta} \leq N\beta^{-\frac{1}{4}} + (\frac{1}{2} + \frac{\varepsilon}{2})\beta^{-\frac{1}{2}} \leq (\frac{1}{2} + \varepsilon)\beta^{-\frac{1}{2}}$$

for $0 < \beta < \beta_0$. Thus, we have $\mathcal{B}_{\beta,N} \subset \{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$ for sufficiently small $\beta > 0$ as long as $K > \frac{1}{2}$. Hence, (3.8) follows once we show

$$\limsup_{\beta \rightarrow 0} |I_f(\mathbf{1}_{\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}}) - I_f(1)| = \limsup_{\beta \rightarrow 0} |I_f(\mathbf{1}_{\mathcal{A}_{\beta,N}^c \cup \mathcal{B}_{\beta,N}^c})| \lesssim N^{-1}. \quad (3.9)$$

By Cauchy-Schwarz inequality along with (3.4), we have

$$|I_f(\mathbf{1}_{\mathcal{A}_{\beta,N}^c \cup \mathcal{B}_{\beta,N}^c})| \leq \left(\mu_{\beta}(\mathcal{A}_{\beta,N}^c \cup \mathcal{B}_{\beta,N}^c) \right)^{\frac{1}{2}} \left(\int e^{6\beta a_{\beta} \int_{\mathbb{T}} u^2} d\mu_{\beta} \right)^{\frac{1}{2}} \lesssim N^{-1} \quad (3.10)$$

since $\int e^{6\beta a_{\beta} \int_{\mathbb{T}} u^2} d\mu_{\beta} = O(1)$ by Lemma 3.2. \square

• (ii) On $\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}^c$: In this case, the Wick-ordered L^4 -norm of u is controlled. Indeed, we have

Lemma 3.5. *For $\beta \leq N^{-4}$, we have*

$$\int_{\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}^c} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \lesssim N^{-2}. \quad (3.11)$$

Proof. From (2.4), we have $\beta^{\frac{1}{2}} a_{\beta} = O(1)$. Thus, on $\mathcal{A}_{\beta,N} \cap \{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$, we have

$$\beta \int_{\mathbb{T}} u^4 \leq \beta \left| \int_{\mathbb{T}} : u^4 :_{\beta} \right| + 6\beta a_{\beta} \int_{\mathbb{T}} u^2 + 3\beta a_{\beta}^2 \lesssim 1$$

for $\beta \leq N^{-4}$. Then, (3.11) follows from (3.4). \square

• (iii) On $\mathcal{A}_{\beta,N}^c$: In this case, we do not have any control on the the Wick-ordered L^4 -norm of u . Nonetheless, we have the following exponential expectation estimate.

Proposition 3.6. *Let $r > 0$. Then, we have*

$$\mathbb{E}_{\mu_{\beta}} \left[\mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int_{\mathbb{T}} u^4} \right] = \int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \leq C(r) < \infty, \quad (3.12)$$

uniformly in small $\beta > 0$.

For each fixed $\beta > 0$, (3.12) follows from [LRS, B2]. The difficulty lies in establishing the estimate uniformly in $\beta > 0$. The proof requires both Fourier analytic and probabilistic approaches. We present the proof of Proposition 3.6 in Sections 4 and 5.

Lemma 3.7. *The following estimate holds uniformly in small $\beta > 0$.*

$$\int_{\mathcal{A}_{\beta,N}^c} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{i \int_{\mathbb{T}} f u + \beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \lesssim N^{-1}. \quad (3.13)$$

Proof. By the Cauchy-Schwarz inequality followed by (3.12) and (3.4), the left hand side of (3.13) is bounded by

$$\left(\int_{\mathcal{A}_{\beta,N}^c} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} d\mu_{\beta} \right)^{\frac{1}{2}} \left(\int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{2\beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \right)^{\frac{1}{2}} \lesssim N^{-1}.$$

□

Finally, (3.1) follows from Lemmas 3.3, 3.4, 3.5, 3.7 by first taking $\beta \rightarrow 0$ and then $N \rightarrow \infty$. Besides proving Proposition 3.6 (which is the content of the next two sections), the only part left is the proof Lemma 3.1 which we present below.

Proof of Lemma 3.1. For any measurable set A , we have

$$\begin{aligned} Q_{0,\beta}^{(4)}(A) &= \frac{\int \mathbf{1}_A \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{\beta \int_{\mathbb{T}} u^4} d\mu_{\beta}}{\int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{\beta \int_{\mathbb{T}} u^4} d\mu_{\beta}} \leq \frac{(\int_A d\mu_{\beta})^{\frac{1}{2}} \left(\int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{2\beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \right)^{\frac{1}{2}}}{\int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{\beta \int_{\mathbb{T}} u^4} d\mu_{\beta}} \\ &\leq C \{\mu_{\beta}(A)\}^{\frac{1}{2}}. \end{aligned} \quad (3.14)$$

In the first line, we used the definition of $Q_{0,\beta}^{(4)}$ and Cauchy-Schwarz inequality. The second line follows from Proposition 3.6 and from the fact that the denominator is bounded from below because of Chebyshev's inequality and Lemma 2.1:

$$\int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{\beta \int_{\mathbb{T}} u^4} d\mu_{\beta} \geq K^{-1} \beta^{\frac{1}{2}} \mathbb{E}_{\mu_{\beta}} \left[\int_{\mathbb{T}} u^2 \right] = K^{-1} \beta^{\frac{1}{2}} a_{\beta} \sim \frac{1}{2} K^{-1} > 0. \quad (3.15)$$

The upper bound (3.14) shows that it is enough to prove that the sequence μ_{β} is tight in $H_0^s(\mathbb{T})$ for $s = -\frac{1}{2} - \varepsilon$, $\varepsilon > 0$. Consider a probability space with the independent standard complex Gaussian random variables g_n with $g_{-n} = \overline{g_n}$. Setting $u^{(\beta)}(x) = \sum_{n \neq 0} \frac{g_n}{\sqrt{1+\beta n^2}} e^{2\pi i n x}$ for $\beta \geq 0$, we have a joint realization of the measures μ_{β} and Q_0 . By the Borel-Cantelli lemma, we have $\sup_{n>0} \frac{|g_n|}{n^{\varepsilon/2}} < \infty$ with probability one. This means that for the Fourier coefficients $\hat{u}_n^{(\beta)}$ of $u^{(\beta)}$, we have $|\hat{u}_n^{(\beta)}| \leq C n^{\varepsilon/2}$ a.s. with a finite (but random) C . Since $\hat{u}_n^{(\beta)} \rightarrow \hat{u}_n^{(0)} = g_n$ a.s. as $\beta \rightarrow 0$ for all n , this implies that $u^{(\beta)} \rightarrow u^{(0)}$ a.s. in $H_0^s(\mathbb{T})$ for $s = -\frac{1}{2} - \varepsilon$. From Prohorov's theorem, we immediately have the tightness of the measures μ_{β} and hence the statement of the lemma. □

4. BOURGAIN'S ARGUMENT: $\lambda > \beta^{-\frac{1}{2}-}$

In this section and next, we present the proof of Proposition 3.6. It follows once we prove the following tail estimate.

Lemma 4.1. *There exists $c, C > 0$ and $\delta > 0$ such that for all $\beta > 0$ and $\lambda \geq 1$,*

$$\mu_{\beta} \left(\beta \|u\|_{L^4(\mathbb{T})}^4 > \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}} \right) \leq C e^{-c\lambda^{1+\delta}} \quad (4.1)$$

We will prove this lemma by considering two cases: $\lambda > \beta^{-\frac{1}{2}-}$ and $\lambda < \beta^{-\frac{1}{2}-}$.³ For fixed $\beta > 0$, Bourgain [B2] proved (4.1) via the dyadic pigeonhole principle with the large deviation estimate (Lemma 4.2.) See Theorem 4.4 below. In this section, we follow his approach to handle the case $\lambda > \beta^{-\frac{1}{2}-}$. For this purpose, we need the following lemma on the tail probabilities of χ^2 random variables.

Lemma 4.2. *Let g_1, g_2, \dots be independent standard real-valued Gaussian random variables. Then for any $M \geq 1$, we have the following large deviation estimate:*

$$P\left[\left(\sum_{n=1}^M g_n^2\right)^{\frac{1}{2}} \geq R\right] \leq e^{-\frac{1}{4}R^2}, \quad R \geq 3M^{\frac{1}{2}}. \quad (4.2)$$

Proof. By Markov's inequality, for $0 \leq t < 1/2$ we have

$$P\left[\left(\sum_{n=1}^M g_n^2\right)^{\frac{1}{2}} \geq R\right] \leq \frac{\mathbb{E}[\exp(t \sum_{n=1}^M g_n^2)]}{\exp(tR^2)} = (1-2t)^{-\frac{M}{2}} e^{-tR^2}.$$

Choosing $t = \frac{1}{2}(1 - \frac{M}{R^2})$, we get the upper bound

$$\left(\frac{R^2}{M}\right)^{\frac{M}{2}} e^{-\frac{1}{2}R^2 + \frac{1}{2}M} \leq e^{\frac{M}{2} \log(R^2/M) + (\frac{1}{18} - \frac{1}{2})R^2} \leq e^{-\frac{1}{4}R^2}$$

where in the last step we used that $\log x \leq x/4$ for $x \geq 9$. \square

Let us introduce some notations. Given $M \in \mathbb{N}$, let $\mathbf{P}_{>M}$ denote the Dirichlet projection onto the frequencies $\{|n| > M\}$. i.e. $\mathbf{P}_{>M}u = \sum_{|n|>M} \hat{u}_n e^{2\pi i n x}$. $\mathbf{P}_{\leq M}$ is defined in a similar manner. Given $j \in \mathbb{N}$, let $M_j = 2^j M$. We use the notation $n \sim M_j$ to denote the set of integers $|n| \in (M_{j-1}, M_j]$, and denote by \mathbf{P}_{M_j} the Dirichlet projection onto the dyadic block $(M_{j-1}, M_j]$ i.e. $\mathbf{P}_{M_j}u = \sum_{n \sim M_j} \hat{u}_n e^{2\pi i n x}$.

Lemma 4.3. *Let $p \geq 2$ and $\beta \leq 1$. Assume that $M \geq \max(\beta^{-\frac{1}{2}-\delta}, \beta^{-\frac{p}{2}+1-\delta})$ for some $\delta > 0$. Then there exists $c, C_1, C_2 > 0$ such that for $\lambda \geq C_1$,*

$$\mu_\beta \left(\beta \|\mathbf{P}_{>M}u\|_{L^p(\mathbb{T})}^p > \lambda \right) \leq C_2 \exp\{-c\lambda^{\frac{2}{p}}\beta^{1-\frac{2}{p}}M^{\frac{2}{p}+1}\} \quad (4.3)$$

Proof. Let $\sigma_j = C2^{-\epsilon j}$, $j = 1, 2, \dots$ for some small $\epsilon > 0$ where $C = C(\epsilon)$ is such that $\sum_{j=1}^{\infty} \sigma_j = 1$. Then, we have

$$\mu_\beta \left(\beta^{\frac{1}{p}} \|\mathbf{P}_{>M_0}u\|_{L^p(\mathbb{T})} > \lambda^{\frac{1}{p}} \right) \leq \sum_{j=0}^{\infty} \mu_\beta \left(\beta^{\frac{1}{p}} \|\mathbf{P}_{M_j}u\|_{L^p(\mathbb{T})} > \sigma_j \lambda^{\frac{1}{p}} \right). \quad (4.4)$$

There is a $c = c(p) < \infty$ such that for all $j = 1, 2, \dots$,

$$\|\mathbf{P}_{M_j}u\|_{L^p(\mathbb{T})} \leq cM_j^{\frac{1}{2}-\frac{1}{p}} \|\mathbf{P}_{M_j}u\|_{L^2(\mathbb{T})}. \quad (4.5)$$

This is the Sobolev inequality, though in this particular case it is a simple application of Hölder's inequality.

From (1.11), we have $\|\mathbf{P}_{M_j}u\|_{L^2(\mathbb{T})}^2 = \sum_{n \sim M_j} |\hat{u}_n|^2 = \sum_{n \sim M_j} (1 + \tilde{\beta}n^2)^{-1} |g_n|^2$. Hence, the right hand side of (4.4) is bounded by

$$\sum_{j=0}^{\infty} P\left[\left(\sum_{n \sim M_j} g_n^2\right)^{1/2} \geq R_j\right], \quad \text{where } R_j := \sigma_j \lambda^{\frac{1}{p}} \beta^{-\frac{1}{p}} M_j^{\frac{1}{p}-\frac{1}{2}} (1 + \beta M_j^2)^{1/2}. \quad (4.6)$$

³We use $a+$ and $a-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for arbitrarily small $\varepsilon \ll 1$.

For $M \geq \max(\beta^{-\frac{1}{2}-\delta}, \beta^{-\frac{p}{2}+1-\delta})$, we have

$$R_j \geq CM^\varepsilon \lambda^{\frac{1}{p}} \beta^{\frac{1}{2}-\frac{1}{p}} M_j^{\frac{1}{p}+\frac{1}{2}-\varepsilon} \geq 3M_j^{\frac{1}{2}}.$$

By Lemma 4.2 to (4.6), we conclude that (4.6) is bounded by $\sum_{j=0}^{\infty} \exp\{-c\sigma_j^2 \lambda^{\frac{2}{p}} \beta^{1-\frac{2}{p}} M_j^{\frac{2}{p}+1}\}$. This completes the proof. \square

Before presenting the proof of Lemma 4.1 for $\lambda > \beta^{-\frac{1}{2}-}$, let us apply Lemma 4.3 to prove the result in [LRS, B2]. Take $\beta = 1$, and let $\mu = \mu_1$.

Theorem 4.4 (Lebowitz, Rose, and Speer [LRS], Bourgain [B2]). *Let $K < \infty$ and $r < \infty$. For $2 < p < 6$, and for $p = 6$ with sufficiently small $K = K(r) > 0$, we have*

$$e^{\int u^p} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\}} \in L^r(d\mu). \quad (4.7)$$

Remark 4.5. The critical value $p = 6$ is related to the L^2 -criticality of the quintic NLS and the quintic generalized KdV.

Proof of Theorem 4.4. It is enough to prove that

$$\int_0^\infty e^\lambda \mu \left(r \int u^p \geq \lambda, \int_{\mathbb{T}} u^2 \leq K \right) d\lambda < \infty.$$

Let $M = c_0 \lambda^{\frac{2}{p-2}} K^{-\frac{p}{p-2}}$ for some $c_0 > 0$. By Sobolev inequality,

$$\|\mathbf{P}_{\leq Mu}\|_{L^p(\mathbb{T})} \leq cM^{\frac{1}{2}-\frac{1}{p}} \|\mathbf{P}_{\leq Mu}\|_{L^2(\mathbb{T})}.$$

Hence, we have $r\|\mathbf{P}_{\leq Mu}\|_{L^p(\mathbb{T})}^p \leq \lambda/2$ on $\int_{\mathbb{T}} u^2 \leq K$. For sufficiently large $\lambda > 0$, the condition of Lemma 4.3 holds, so we have

$$\mu \left(r\|\mathbf{P}_{>Mu}\|_{L^p(\mathbb{T})}^p > \lambda \right) \leq C \exp\{-cr^{-\frac{p}{2}} \lambda^{\frac{2}{p}} M^{\frac{2}{p}+1}\} = C \exp\{-c'\lambda^{1+\frac{6-p}{p-2}} r^{-\frac{p}{2}} K^{-\frac{p+2}{p-2}}\}. \quad (4.8)$$

and the statement follows. Note that when $p = 6$, we need to take $K = K(r)$ sufficiently small such that $r^{-3}K^{-2}$ is large and the coefficient of λ is less than -1 in (4.8). \square

Now, we present the proof of Lemma 4.1 for $\lambda > \beta^{-\frac{1}{2}-}$. As we see, one obtains much less in estimating the tail uniformly in $\beta > 0$ even when $p = 4$. Indeed, Bourgain's argument is not enough to conclude the argument even for $p = 3$.

Proof of Lemma 4.1 for $\lambda > \beta^{-\frac{1}{2}-}$. The proof is similar to that of Theorem 4.4. First, choose $M = c_0 K^{-2} \lambda > \beta^{-\frac{1}{2}-}$. By Sobolev inequality,

$$\beta \|\mathbf{P}_{\leq Mu}\|_{L^4(\mathbb{T})}^4 \leq c\beta M \|\mathbf{P}_{\leq Mu}\|_{L^2(\mathbb{T})}^4.$$

Hence, on $\|\mathbf{P}_{\leq Mu}\|_{L^2(\mathbb{T})} \leq K^{\frac{1}{2}} \beta^{-\frac{1}{4}}$, we have, for sufficiently small c_0 ,

$$\beta \|\mathbf{P}_{\leq Mu}\|_{L^4(\mathbb{T})}^4 \leq c c_0^4 \lambda \leq \lambda/2.$$

As before, we can apply Lemma 4.2 to handle the high frequencies as long as $R_j \geq 3M_j^{\frac{1}{2}}$ in (4.6). Unlike the proof of Lemma 4.3, when checking this, we use the non-smallness of $M_j \geq \lambda > \beta^{-\frac{1}{2}-}$. In this case, we have

$$R_j = \sigma_j \lambda^{\frac{1}{4}} \beta^{-\frac{1}{4}} M_j^{-\frac{1}{4}} (1 + \beta M_j^2)^{1/2} \geq \beta^{\frac{1}{8}-} M_j^{\frac{3}{4}-\varepsilon} \geq M_j^{\frac{1}{2}+}.$$

By proceeding as in the proof of Lemma 4.3, we obtain (4.3). Then, (4.1) follows once we note that $M_j \gtrsim \lambda > \beta^{-\frac{1}{2}-}$. \square

5. HYPERCONTRACTIVITY ESTIMATE: $\lambda < \beta^{-\frac{1}{2}-}$

First, note that we have $\beta \int_{\mathbb{T}} u^4 = \beta \int :u^4:_{\beta} + O(1)$ on $\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$ and thus it is enough to prove (4.1) with $\beta \int :u^4:_{\beta}$ instead of $\beta \int_{\mathbb{T}} u^4$. We will use the identity (2.10) and we further separate the summation for Π into (a) n_j all distinct, (b) $n_1 = n_2 \neq n_3, n_4$ and $n_3 \neq n_4$, and (c) $n_1 = n_2 = n_3 \neq n_4$ (up to permutations of the indices) and write $\Pi = \Pi_a + \Pi_b + \Pi_c$. Recall also the definitions of I_1 and I_2 from (2.10). We will show that the main contribution of $\beta \int :u^4:_{\beta}$ comes from “no pair, all distinct”, i.e. Π_a .

Lemma 5.1. *On $\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$, there is a $C < \infty$ such that $\beta|I_1|, \beta|I_2|, \beta|\Pi_b|, \beta|\Pi_c| \leq C$ uniformly in $\beta > 0$.*

Proof. In view of (1.11), we have

$$\beta|I_1| = \beta \left(\sum_{n \geq 1} \frac{|g_n|^2 - 1}{1 + \tilde{\beta}n^2} \right)^2 \leq 2\beta \left(\sum_{n \geq 1} \frac{|g_n|^2}{1 + \tilde{\beta}n^2} \right)^2 + 2\beta \left(\sum_{n \geq 1} \frac{1}{1 + \tilde{\beta}n^2} \right)^2 \lesssim 1$$

on $\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$. By Hölder inequality and $l^2 \subset l^4$, the contribution for Π from the case (c) is at most

$$\begin{aligned} \beta|\Pi_c| &\sim \beta \left| \sum_{n_1 \neq 0} \frac{g_{n_1}^3}{(1 + \tilde{\beta}n_1^2)^{\frac{3}{2}}} \frac{g_{-3n_1}}{\sqrt{1 + \tilde{\beta}(-3n_1)^2}} \right| \leq \beta \sum_{n \neq 0} \frac{|g_n|^4}{(1 + \tilde{\beta}n^2)^2} \\ &\leq \beta \left(\sum_{n \neq 0} \frac{|g_n|^2}{1 + \tilde{\beta}n^2} \right)^2 = \beta \left(\int_{\mathbb{T}} u^2 \right)^2 \lesssim 1. \end{aligned}$$

Similarly, we have $\beta|I_2| \lesssim 1$. Then, the contribution for Π from the case (b) is at most

$$\begin{aligned} \beta|\Pi_b| &\sim \beta \left| \sum_{\substack{\text{no pair} \\ n_1, n_3 \neq 0}} \frac{g_{n_1}^2}{1 + \tilde{\beta}n_1^2} \frac{g_{n_3}}{\sqrt{1 + \tilde{\beta}n_3^2}} \frac{g_{-2n_1-n_3}}{\sqrt{1 + \tilde{\beta}(-2n_1-n_3)^2}} \right| \\ &\leq \beta \sum_{n_1 \neq 0} \frac{|g_{n_1}|^2}{1 + \tilde{\beta}n_1^2} \sup_{n_1 \neq 0} \left| \sum_{n_3 \neq 0} \frac{g_{n_3}}{\sqrt{1 + \tilde{\beta}n_3^2}} \frac{g_{-2n_1-n_3}}{\sqrt{1 + \tilde{\beta}(-2n_1-n_3)^2}} \right| \\ &\leq \beta \left(\sum_{n \neq 0} \frac{|g_n|^2}{1 + \tilde{\beta}n^2} \right)^2 = \beta \left(\int_{\mathbb{T}} u^2 \right)^2 \lesssim 1, \end{aligned}$$

where we used $ab \leq a^2/2 + b^2/2$ in the last line. \square

In estimating the contribution from Π_a = “no pair, all distinct”, we will use the hypercontractivity of the Ornstein-Uhlenbeck process. Let L denote the generator of the Ornstein-Uhlenbeck process on $H := L^2(\mathbb{R}^d, e^{-|x|^2/2} dx)$ given by $L = \Delta - x \cdot \nabla$. Then, let $S(t) = \exp(tL)$ be the semigroup associated with $\partial_t u = Lu$. Then, the hypercontractivity of the Ornstein-Uhlenbeck semigroup [T, Sec.3] says the following:

Lemma 5.2. *Let $q \geq 2$. For $f \in H$ and $t \geq \frac{1}{2} \log(q-1)$, we have*

$$\|S(t)f\|_{L^q(\mathbb{R}^d, \exp(-|x|^2/2)dx)} \leq \|f\|_{L^2(\mathbb{R}^d, \exp(-|x|^2/2)dx)}$$

The eigenfunctions of L are given by $\prod_{j=1}^d h_{k_j}(x_j)$, where h_k is the Hermite polynomial of degree k , and the corresponding eigenvalue is given by $\lambda = -(k_1 + \dots + k_d)$. The first

few Hermite polynomials are

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad \dots$$

Let

$$H(x) = \sum_{\Gamma} c(n_1, \dots, n_4) x_{n_1} \cdots x_{n_4},$$

where $\Gamma = \{(n_1, \dots, n_4) \in \{1, \dots, d\}^4, \text{ all distinct}\}$. Note that $H(x)$ is an eigenfunction of L with the eigenvalue -4 . The following *dimension-independent* estimate is a simple consequence of Lemma 5.2:

Corollary 5.3. *For all $d = 1, 2, 3, \dots$, we have*

$$\|H(x)\|_{L^q(\mathbb{R}^d, \exp(-|x|^2/2)dx)} \leq q^2 \|H(x)\|_{L^2(\mathbb{R}^d, \exp(-|x|^2/2)dx)}. \quad (5.1)$$

Proof of Lemma 4.1 for $\lambda < \beta^{-\frac{1}{2}-}$. By Lemma 5.1 and the argument just preceding it, all it suffices to prove

$$\mu_\beta\left(|\Pi_a| \geq \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\right) \leq Ce^{-c\lambda^{1+\delta}} \quad (5.2)$$

for $\lambda \leq \beta^{-\frac{1}{2}-}$. First, we show

$$\mu_\beta\left(|F_{\beta,M}| \geq \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\right) \leq Ce^{-c\lambda^{1+\delta}} \quad (5.3)$$

for $\lambda \leq \beta^{-\frac{1}{2}-}$, where

$$F_{\beta,M} = \beta \sum_{**} \prod_{j=1}^4 \frac{g_{n_j}}{\sqrt{1 + \tilde{\beta} n_j^2}} \quad (5.4)$$

with $** = \{n_{1234} := n_1 + \dots + n_4 = 0, n_j \neq 0, \text{ no pair, all distinct, } |n_j| \leq M\}$, with a constant c independent of M . Then, we will indicate how (5.2) follows from (5.3).

By expanding the complex-valued Gaussians g_n into their real and imaginary parts, we can apply (5.1) to $Q_{\beta,M}$ in (5.4). From (the proof of) Lemma 2.1, we have $\|F_{\beta,M}\|_{L^2(d\mu_\beta)} \leq C\beta^{\frac{1}{4}}$. By (5.1), we have

$$\|F_{\beta,M}\|_{L^q(d\mu_\beta)} \leq Cq^2\beta^{\frac{1}{4}} \quad (5.5)$$

for all $q \geq 2$. Note that we need that u has a finite Fourier support, but the actual upperbound on the support is not important. Then, we have

$$\int \exp(c\beta^{-\frac{1}{8}}|F_{\beta,M}|^{\frac{1}{2}})d\mu_\beta \leq C \quad (5.6)$$

from Lemma 4.5 in [T]. This can be proved by expanding the exponential in the Taylor series and applying (5.5) and Hölder's inequality. Equation (5.6) in turn implies $\mu_\beta(|F_{\beta,M}| > \lambda) \leq C \exp(-c'\beta^{-\frac{1}{8}}\lambda^{\frac{1}{2}})$ by Markov's inequality, i.e. we proved (5.3) for $\lambda \leq \beta^{-\frac{1}{4}+}$.

Now, we consider the remaining case: $\beta^{-\frac{1}{4}+} \leq \lambda \leq \beta^{-\frac{1}{2}-}$. Then, using $\lambda \geq \beta^{-\frac{1}{4}+\varepsilon}$,

$$\mu_\beta(|F_{\beta,M}| \geq \lambda) \leq \frac{\|F_{\beta,M}\|_{L^q(d\mu_\beta)}^q}{\lambda^q} \leq Cq^{2q}\beta^{\frac{q}{2}-\varepsilon q} \leq e^{2q \ln q} e^{-\frac{q}{3} \ln \beta^{-1}} = e^{-\frac{q}{3} \ln \beta^{-1} + 2q \ln q}$$

By choosing $q \sim \beta^{-\frac{3}{4}} \ll \beta^{-1}$ and using $\lambda \leq \beta^{-\frac{1}{2}-\varepsilon}$,

$$\leq e^{-c\beta^{-\frac{3}{4}} \ln \beta^{-1}} \leq e^{-c\lambda^{\frac{3}{2}-}}.$$

This proves (5.3).

Now, we need to show how (5.2) follows from (5.3). Clearly, $F_{\beta,M} \rightarrow \Pi_a$ in $L^2(d\mu_\beta)$ as $M \rightarrow \infty$. Thus, we can find a subsequence $M_k \rightarrow \infty$ for which $F_{\beta,M_k} \rightarrow \Pi_a$ almost surely with respect to μ_β . By the dominated convergence theorem for the indicator random variables $\mathbf{1}\left(|F_{\beta,M_k}| \geq \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\right)$, we have, for fixed $\beta > 0$ and $\lambda \geq 1$,

$$\mu_\beta\left(|\Pi_a| \geq \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\right) = \lim_{k \rightarrow \infty} \mu_\beta\left(|F_{\beta,M_k}| \geq \lambda, \int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\right) \leq Ce^{-c\lambda^{1+\delta}},$$

where C and c are independent of β and λ . This completes the proof of the tail estimate (4.1). \square

6. REMARKS

We proved Theorem 1.1 for $p = 4$. In this section, we briefly discuss the minor changes needed to handle the complex-valued case, the focusing case (Theorem 1.3), and the $p = 3$ case.

• **Complex-valued case:** As mentioned in Remark 1.2, the same result holds for the complex-valued case as well. In this case, one needs to use the following definitions of Wick-ordered monomials,

$$\begin{aligned} :|u|^2:_\beta &= |u|^2 - a_\beta, \\ :|u|^4:_\beta &= |u|^4 - 4a_\beta|u|^2 + 2a_\beta^2, \end{aligned}$$

where $a_\beta = \mathbb{E}_{\mu_\beta}[\int_{\mathbb{T}} |u|^2]$. The proof is basically the same (note that we did not really need the mean-zero condition), and one needs to prove Proposition 3.6 in the complex-valued case. This follows easily once we note $|u|^4 \lesssim (\operatorname{Re} u)^4 + (\operatorname{Im} u)^4$.

• **Defocusing case:** Now, let us briefly discuss the proof of Theorem 1.3. First, write

$$\begin{aligned} \int e^{i \int f u} d\tilde{Q}_{0,\beta}^{(4)} &= Z_\beta^{-1} \int e^{i \int f u - \beta \int_{\mathbb{T}} u^4} d\mu_\beta \\ &= Z_\beta^{-1} \int e^{i \int f u - \beta \int_{\mathbb{T}} u^4} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} d\mu_\beta + Z_\beta^{-1} \int e^{i \int f u - \beta \int_{\mathbb{T}} u^4} \mathbf{1}_{\{\int_{\mathbb{T}} u^2 > K\beta^{-\frac{1}{2}}\}} d\mu_\beta. \end{aligned}$$

By repeating the argument in Section 3, the first term yields the desired result. Note that we have Proposition 3.6 for free thanks to the negative sign. As for the second term, (3.4) states that the contribution on $\mathcal{A}_{\beta,N}^c \cup \mathcal{B}_{\beta,N}^c$ goes to 0 as $N \rightarrow \infty$. The contribution on $\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N}$ also goes to 0 since $\mathcal{A}_{\beta,N} \cap \mathcal{B}_{\beta,N} \subset \mathcal{B}_{\beta,N} \subset \{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$ for sufficiently small $\beta > 0$ for $K > \frac{1}{2}$.

Note that Lemma 3.1 follows in a similar manner as before, once we show that the denominator in (3.14) is bounded from below. By Jensen's inequality we have

$$\int_{\mathcal{A}} e^{-\beta \int_{\mathbb{T}} u^4} d\mu_\beta \geq \mu_\beta(\mathcal{A}) \exp \left\{ -\frac{1}{\mu_\beta(\mathcal{A})} \mathbb{E}_{\mu_\beta} \left[\mathbf{1}_{\mathcal{A}} \beta \int_{\mathbb{T}} u^4 \right] \right\} \quad (6.1)$$

where $\mathcal{A} = \{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}$. The right hand side is clearly bounded from below as $\beta \rightarrow 0$ since $\beta \int_{\mathbb{T}} u^4 \rightarrow C$ by Lemma 2.1 and $\mu_\beta(\mathcal{A})$ is bounded from below by Chebyshev (c.f. (3.15)).

• **$p = 3$ case:** The proof of Theorem 1.1 for $p = 3$ is similar to the $p = 4$ case. Once we have Lemma 4.1, everything follows for $p < 4$. However, in this case, we do not need to use the Wick-ordered $\int_{\mathbb{T}} u^3$, and a simpler proof is available because the hypercontractivity

estimates can be replaced by a direct application of the Sobolev inequality, but it is still a nontrivial extension of the Bourgain method. We sketch it now.

By direct computation, we have

$$\mathbb{E}_{\mu_\beta} \left[\int_{\mathbb{T}} u^3 \right] = 0, \quad \text{and} \quad \mathbb{E}_{\mu_\beta} \left[\left(\int_{\mathbb{T}} u^3 \right)^2 \right] \lesssim \beta^{-1}.$$

Similarly to the $p = 4$ case we define $\mathcal{C}_{\beta,N}$ by

$$\mathcal{C}_{\beta,N} = \left\{ \left| \int_{\mathbb{T}} u^3 \right| \leq N\beta^{-\frac{1}{2}} \right\}, \quad (6.2)$$

and separately estimate the contributions from

$$(i) \mathcal{A}_{\beta,N} \cap \mathcal{C}_{\beta,N}, \quad (ii) \mathcal{A}_{\beta,N} \cap \mathcal{C}_{\beta,N}^c, \quad \text{and} \quad (iii) \mathcal{A}_{\beta,N}^c.$$

The main contribution comes from $\mathcal{A}_{\beta,N} \cap \mathcal{C}_{\beta,N}$. Unlike the $p = 4$ case, there is no need to introduce $\tilde{\mu}_\beta$ defined in (3.5), and we can simply use the convergence of μ_β : $\lim_{\beta \rightarrow 0} \int e^{i \int_{\mathbb{T}} f u} d\mu_\beta = e^{-\frac{1}{2} \|f\|_2^2}$ for any mean zero smooth function f on \mathbb{T} .

The contributions from $\mathcal{A}_{\beta,N} \cap \mathcal{C}_{\beta,N}^c$ and $\mathcal{A}_{\beta,N}^c$ can be shown to be small by Chebyshev's inequality, once we prove the following exponential expectation bound.

Proposition 6.1. *Let $r > 0$. Then, we have*

$$\mathbb{E}_{\mu_\beta} \left[\mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int_{\mathbb{T}} u^3} \right] = \int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int_{\mathbb{T}} u^3} d\mu_\beta \leq C(r) < \infty, \quad (6.3)$$

uniformly in small $\beta > 0$.

Proposition 6.1 is a corollary of Proposition 3.6. However, there is an easier direct proof in this case:

Proof. By Sobolev inequality followed by Hölder's inequality, we have

$$\int_{\mathbb{T}} u^3 \leq c \left(\sum_{n \neq 0} n^{\frac{1}{3}} |\hat{u}_n|^2 \right)^{\frac{3}{2}} \leq c \left(\sum_{n \neq 0} n^{\frac{1}{2}} |\hat{u}_n|^2 \right) \left(\sum_{n \neq 0} |\hat{u}_n|^2 \right)^{1/2} \leq cK^{\frac{1}{2}} \beta^{-\frac{1}{4}} \sum_{n \neq 0} n^{\frac{1}{2}} |\hat{u}_n|^2$$

on $\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}$. Moreover, we have

$$\beta \int_{\mathbb{T}} [\mathbf{P}_{\leq c_0 \beta^{-\frac{1}{2}}} u]^3 \lesssim \beta^{\frac{3}{4}} \sum_{1 \leq |n| \leq c_0 \beta^{-\frac{1}{2}}} n^{\frac{1}{2}} |\hat{u}_n|^2 \leq C$$

on $\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}$. Hence, from (1.11), we have

$$\begin{aligned} \int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} e^{r\beta \int_{\mathbb{T}} u^3} d\mu_\beta &\leq \int \mathbf{1}_{\{\int_{\mathbb{T}} u^2 \leq K\beta^{-\frac{1}{2}}\}} \exp \left\{ C + c\beta^{\frac{3}{4}} \sum_{|n| > c_0 \beta^{-\frac{1}{2}}} n^{\frac{1}{2}} |\hat{u}_n|^2 \right\} d\mu_\beta \\ &\lesssim \int \prod_{n > c_0 \beta^{-\frac{1}{2}}} \exp \left\{ \frac{2c\beta^{\frac{3}{4}} n^{\frac{1}{2}}}{1 + \tilde{\beta} n^2} |g_n|^2 \right\} d\mu_\beta = \prod_{n > c_0 \beta^{-\frac{1}{2}}} \frac{1}{1 - \frac{c\beta^{\frac{3}{4}} n^{\frac{1}{2}}}{1 + \tilde{\beta} n^2}}, \end{aligned} \quad (6.4)$$

where in the last equality we used $\mathbb{E}[e^{aX^2}] = (1 - 2a)^{-\frac{1}{2}}$, $a < \frac{1}{2}$ for a real valued standard Gaussian random variable X , since $(c\beta^{\frac{3}{4}} n^{\frac{1}{2}})(1 + \tilde{\beta} n^2)^{-1} < \frac{1}{2}$ on $n > c_0 \beta^{-\frac{1}{2}}$ for sufficiently large $c_0 > 0$.

It is not hard to check that $0 < x < 1/2$ implies $(1-x)^{-1} < e^{x+x^2}$.

$$(6.4) \leq \prod_{n > c_0 \beta^{-\frac{1}{2}}} \exp \left\{ \frac{c \beta^{\frac{3}{4}} n^{\frac{1}{2}}}{1 + \tilde{\beta} n^2} + \frac{c^2 \beta^{\frac{3}{2}} n}{(1 + \tilde{\beta} n^2)^2} \right\} = \exp \left\{ \sum_{n > c_0 \beta^{-\frac{1}{2}}} \frac{c \beta^{\frac{3}{4}} n^{\frac{1}{2}}}{1 + \tilde{\beta} n^2} + \frac{c^2 \beta^{\frac{3}{2}} n}{(1 + \tilde{\beta} n^2)^2} \right\}.$$

Hence, by Riemann sum approximation, we have for sufficiently small $\beta > 0$,

$$\sum_{n > c_0 \beta^{-\frac{1}{2}}} \frac{\beta^{\frac{3}{4}} n^{\frac{1}{2}}}{1 + \tilde{\beta} n^2} + \frac{\beta^{\frac{1}{2}} n}{(1 + \tilde{\beta} n^2)^2} \lesssim \int_{c_0}^{\infty} \frac{\sqrt{x}}{1+x^2} dx + \beta^{1/2} \int_{c_0}^{\infty} \frac{x}{(1+x^2)^2} dx < \infty.$$

This shows that (6.4) is finite. \square

Lastly, note that Lemma 3.1 follows as before, once we show that the denominator in (3.14) (with $p = 3$) is bounded from below. Proceeding the same way as in (6.1) this is immediate since $\mathbb{E}_{\mu_\beta} \left[\mathbf{1}_{\int_{\mathbb{T}} u^2 \leq K \beta^{-\frac{1}{2}}} \beta \int_{\mathbb{T}} u^3 \right] = 0$ by the $u \rightarrow -u$ symmetry of μ_β .

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